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# An Abraham–Lorentz-like equation for the electron from the worldline variational approach to QED

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**Abstract.** The variational equation for the mean square displacement of the electron in the polaron worldline approach to quenched QED can be cast into a form which closely resembles the classical Abraham–Lorentz equation but without the conceptual and practical diseases of the latter. The connection with delay equations describing field retardation effects is also established. As applications we solve this integro-differential equation numerically for various values of the coupling constant and cut-off and re-derive the variational approximation to the anomalous mass dimension of the electron found recently.

## 1 Introduction

There is a long history of attempts at classical models of the electron until quantum electrodynamics (QED) took over as the most successful and precise theory in the microscopic domain [1]. Noteworthy among these models is, in particular, the one due to Abraham and Lorentz nearly one hundred years ago which is covered in many textbooks (see, for example, [2]). It describes a classical electron under the influence of both an external force as well as the back reaction from the energy loss due to radiation of photons, and leads to a third-order differential equation which, however, has extraneous unphysical *run-away* solutions. The usual method to avoid these solutions is to convert the original equation into a second-order integro-differential equation. The classic non-relativistic result is of the well-known form [2]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{x}(t) \;=\; \frac{1}{m}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s}\,\mathbf{F}(t+s\tau) \quad , \qquad (1)$$

where $\tau$ is the characteristic time $2e^2/3mc^3$, $m$ is the electron mass and $\mathbf{F}$ is an external force which can depend on time either explicitly or implicitly through $\mathbf{x}(t)$. The solutions to this equation do not exhibit run-away behavior, however (1) clearly violates (non-relativistic) causality because the motion at time $t$ depends on forces at time $t' > t$. It is noteworthy that the analogous equation, without runaway solutions, for a relativistic Dirac particle [3] in 3 dimensions has only been derived rather recently [4].

From a modern perspective, of course, it is appropriate to wonder how non-relativistic and classical equations like the one above are connected to a quantum mechanical and ultimately a quantum field-theoretical description. In non-relativistic QED this correspondence limit has been examined in particular by Moniz and Sharp [5] while Johnson and Hu have derived an Abraham–Lorentz–Dirac equation as a semi-classical limit within scalar quantum field theory [6].

In the present paper we discuss the fully relativistic field-theoretic problem of a spin-1/2 electron dressed by an arbitrary number of photons in the quenched approximation to QED. Our approach is patterned after Feynman's celebrated variational treatment of the polaron problem [7], which was first applied by Mano [8] to a relativistic scalar field theory and re-discovered and expanded by us in a series of papers [9,10]. Its main features are the description of relativistic particles by worldlines [11] parametrized by the proper time, an exact functional integration over the light fields (i.e. photons) and a variational approximation of the resulting effective action by a retarded quadratic trial action. In recent work we have extended this approach to more realistic theories, in particular to quenched QED [12,13].

The approach is fully non-perturbative, Lorentz covariant, respects gauge symmetry and contains the exact one-loop self-energy in the small coupling limit. It does not rely on a perturbative, semi-classical or a derivative expansion. In [13] we concentrated on the divergence structure and renormalization of the theory, resulting in a compact ex-

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pression for the anomalous mass dimension of the electron. In the present work we go beyond this and calculate also the finite contributions. We do this by numerically solving the variational equation for the position of the pole of the electron propagator for a range of coupling constants and UV cut-offs. We shall also show that the relevant variational equation can be written as an Abraham–Lorentz-like equation for the *mean square* displacement of the electron. As this "variational Abraham–Lorentz equation" (VALE) turns out to be (at most) a second-order integro-differential equation, no run-away solutions occur. Indeed, anticipating the result derived in Sect. 3, in general this equation may be written in the suggestive form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\langle x^2(t)\right\rangle = 2\left\langle \dot{x}^2(t)\right\rangle + \frac{2}{\kappa_\mathrm{E}}\int_{-\infty}^{+\infty}\mathrm{d}s\, f(s)\left\{\left\langle [\Delta x(t)]^2\right\rangle - \left\langle [\Delta x(t-s)]^2\right\rangle\right\}. \quad (2)$$

Here $x_\mu(t)$ is the Euclidean position at proper time $t$ ($t>0$), $\Delta x^\mu(t)$ is its deviation from a classical straight-line path, $\kappa_\mathrm{E}$ is a reparameterization parameter playing the role of a mass and $\langle\ldots\rangle$ refers to a certain averaging over all worldlines occurring in the functional integral for the theory. These definitions will be made more precise in the following two sections. At this stage we merely point out the marked similarity of this equation to (1), in particular if one re-writes the latter as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{x}^2(t) \;=\; 2\,\dot{\mathbf{x}}^2(t) \;+\; \frac{2}{m}\,\mathbf{x}(t)\cdot\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s}\mathbf{F}(t+s\tau)\,. \quad (3)$$

Of course, (2) and (3) are not completely identical, nor should they be. In the absence of external forces the latter equation yields a differential equation with the solution that the particle moves uniformly along a straight line. On the other hand, there *are no* external forces in (2). The first term on the RHS is the classical term while the integral characterizes an "internal force" which has its origin in the constant emission and re-absorption of virtual photons. This term is sensitive only to the (random-walk-like) deviations from a straight line.

In the next section we will describe the derivation of the variational equations, leading to the VALE (2) in Sect. 3 and will solve it numerically in Sect. 4. Understanding these numerical results will also give a new derivation of the variational approximation to the anomalous mass dimension, simpler than the one presented in [13]. Approximate forms of the VALE are discussed in an appendix.

## 2 The variational method

In order to introduce our notation and terminology, we begin with a brief summary of the essential points of the variational technique applied to quenched QED and refer the reader to [9,12] and in particular [13] for the details.

### 2.1 Worldline formulation

The starting point is the worldline, rather than field-theoretic, description for the propagator of a spinning particle which involves functional integrals over a bosonic worldline $x_\mu(t)$ as well as a Grassmannian $\zeta_\mu(t)$ which characterizes the spin. In Minkowski space time (with metric $(+,---)$) the free part (modulo boundary terms) reads

$$S_0 \;=\; \int_0^T \mathrm{d}t\;\left[-\frac{\kappa_0}{2}\,\dot{x}^2 + \mathrm{i}\,\zeta\cdot\dot{\zeta} + \frac{1}{T}\,\dot{x}\cdot\zeta\chi\right], \quad (4)$$

where $\kappa_0$ is the reparametrization parameter and $\chi$ the supersymmetric (SUSY) counterpart to the proper time $T$. The dependence of the action on these two degrees of freedom is connected by a supersymmetric transformation [12]. Due to time-translational invariance only the time interval $T$ matters and therefore in reality the integration limits in (4) and all following expressions are $[t_0, T+t_0]$ with $t_0$ arbitrary (e.g. 0 or $-T/2$). The Gaussian functional integral over the photon field in the interaction part can be carried out analytically, resulting in a bi-local effective action

$$S_1 = -\frac{e^2}{2}\int_0^T \mathrm{d}t_1\mathrm{d}t_2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,G^{\mu\nu}(k)\,J_\mu(k,x_1,\zeta_1) \times J_\nu(-k,x_2,\zeta_2)\,\mathrm{e}^{-\mathrm{i}k\cdot(x_1-x_2)}, \quad (5)$$

where $G^{\mu\nu}(k)$ is the photon propagator in an arbitrary (covariant) gauge,

$$J_\mu(k,x,\zeta) \;=\; \dot{x}_\mu - \frac{2}{\kappa_0}\,\zeta_\mu\,k\cdot\zeta \quad (6)$$

is the (convection and spin) current of the electron and $x_1 \equiv x(t_1)$ etc. There is an elegant, compact and manifestly supersymmetric formulation in terms of "superpositions" and "superderivatives" [12] but in the following we will use the more cumbersome but explicit decomposition into bosonic and fermionic worldlines.

### 2.2 Feynman–Jensen stationarity and variational principle

Having integrated out the photons, the functional integrals over $x$ and $\zeta$ cannot be performed exactly but may be approximated via the introduction of a trial action $S_\mathrm{t}[x,\zeta]$ and by making use of Feynman's variational technique:

$$\int \mathrm{e}^{\mathrm{i}S} = \int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}\cdot\frac{\int \mathrm{e}^{\mathrm{i}S}}{\int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}} \;=\; \int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}\cdot\frac{\int \mathrm{e}^{\mathrm{i}(S-S_\mathrm{t})}\,\mathrm{e}^{\mathrm{i}S_\mathrm{t}}}{\int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}} \equiv \int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}\cdot\left\langle \mathrm{e}^{\mathrm{i}(S-S_\mathrm{t})}\right\rangle_{S_\mathrm{t}} \simeq \int \mathrm{e}^{\mathrm{i}S_\mathrm{t}}\cdot\mathrm{e}^{\mathrm{i}\langle S-S_\mathrm{t}\rangle_{S_\mathrm{t}}}. \quad (7)$$

Even with Euclidean times, where $\exp(\mathrm{i}S)\to\exp(-S_\mathrm{E})$, the usual Jensen inequality does not hold anymore since we are also integrating over Grassmann-valued trajectories. However, only stationarity of the above expression

under variations of the trial action is required in the following. The subscript $S_\mathrm{t}$ on the average reminds us that the weight function is $\mathrm{e}^{\mathrm{i}S_\mathrm{t}}$; the averages in (2) are to be understood in the same way. With an arbitrary trial action the above approximation becomes exact at the stationary point. In practise, however, only trial actions at most quadratic in $x$ and $\zeta$ can be used. We work with a trial action of this form obtained by making the modified free action $\tilde{S}_0 = S_0 + p\cdot x$ bi-local[1]

$$\tilde{S}_\mathrm{t} = \lambda_1\, p\cdot x + \int_0^T \mathrm{d}t_1\,\mathrm{d}t_2\,\Big[-\frac{\kappa_0}{2}\,g_B(\sigma)\,\dot{x}_1\cdot\dot{x}_2 + \mathrm{i}\,g_F'(\sigma)\,\zeta_1\cdot\zeta_2 - \frac{1}{T}\,\sigma\,g_M'(\sigma)\,\dot{x}_1\cdot\zeta_2\chi\Big]\ . \quad (8)$$

In addition to a scalar variational parameter $\lambda_1$ this trial action contains three arbitrary (but even) variational "retardation" functions of $\sigma = t_1 - t_2$ multiplying the three quadratic combinations $\dot{x}\,\dot{x}$ (i.e. "bosonic", or $B$), $\zeta\,\zeta$ ("fermionic", $F$) and $\dot{x}\,\zeta$ ("mixed", $M$). Only the first two of these are relevant in the calculation of the pole position of the electron propagator, which is obtained for $T\to\infty$. An explicitly supersymmetric trial action would require only one retardation function,

$$g_B(\sigma) \;=\; g_F(\sigma) \;=\; g_M(\sigma) \;\equiv\; g(\sigma)\quad . \quad (9)$$

As the exact action contains SUSY-breaking boundary terms (which will be discussed in more detail elsewhere [14]) we shall not enforce (9). It is therefore advisable to take the most general ansatz and let the variational principle choose the optimal solution within the given class of test functions[2]. Note that for $\lambda_1 = 1, g_B(\sigma) = g_F(\sigma) = g_M(\sigma) = \delta(\sigma)$ the trial action (8) reduces to the free action (4). Because averaging with the free action is equivalent to first-order perturbation theory this implies that the variational approach gives the correct one-loop self-energy for small coupling.

### 2.3 Mano's equation

The averages involved in the last term of (7) may be separated into averages of $S_0 - S_\mathrm{t}$ (which, together with $\int\exp(\mathrm{i}S_\mathrm{t})$, may be combined into a quantity $\Omega$) and the average of $S_1$ (denoted by $V$). The roles of $\Omega$ and $V$ are not unlike those of, respectively, kinetic and potential contributions in a standard quantum mechanical variational calculation. Near the pole the electron propagator takes the form

$$G_2(p) \;\longrightarrow\; Z_2\,\frac{\not{p}+M}{p^2-M^2}, \quad (10)$$

and the on-shell limit of the argument of the exponential of the last term in (7) directly yields the relationship between the electron's bare mass $M_0$ and its physical mass $M$. Explicitly, this equation, termed "Mano's equation", becomes

$$M_0^2 = M^2(2\lambda-\lambda^2) - 2(\Omega[A_B] - \Omega[A_F] + V[\mu_B^2,\mu_F^2]). \quad (11)$$

Here the "profile functions" $A_B$ and $A_F$ are Fourier transforms of the bosonic and fermionic variational retardation functions, respectively:

$$A_i(E) = \int_{-\infty}^{+\infty}\mathrm{d}\sigma\; g_i(\sigma)\mathrm{e}^{\mathrm{i}E\sigma} = 2\int_0^\infty \mathrm{d}\sigma\; g_i(\sigma)\cos(E\sigma)\ , \qquad i\;=\;B,\,F\,. \quad (12)$$

They are fixed, as is $\lambda \equiv \lambda_1/A_B(0)$, through the Feynman–Jensen variational principle which guarantees that Mano's equation (11) is stationary with respect to their variation. Once fixed, all quantities of interest in the field theory (e.g. masses, form factors, scattering cross sections [9]) may be expressed in terms of these. $A_B$ and $A_F$ only appear implicitly in the potential $V$, through the bosonic and fermionic "pseudotimes" $\mu^2_{B,F}(\sigma)$ defined by

$$\mu_i^2(\sigma) \;=\; \frac{4}{\pi}\int_0^\infty \mathrm{d}E\;\frac{1}{E^2A_i(E)}\;\sin^2\left(\frac{E\sigma}{2}\right). \quad (13)$$

### 2.4 Kinetic and potential terms

Explicitly, the kinetic terms $\Omega[A_i]$ are given, in 4-dimensional Euclidean space, by

$$\Omega[A_i] \;=\; \frac{2\kappa_\mathrm{E}}{\pi}\int_0^\infty \mathrm{d}E\left(\log A_i(E) + \frac{1}{A_i(E)} - 1\right). \quad (14)$$

Here $\kappa_\mathrm{E} > 0$ is a parameter which re-parameterizes the proper time without changing the physics. It is useful to keep it because it plays the role of a mass in the worldline description (see footnote 3 in [10]; the Euclidean parameter $\kappa_\mathrm{E}$ is related to its Minkowski counterpart $\kappa_0$ through $\kappa_0 = \mathrm{i}\kappa_\mathrm{E}$).

The potential term $V$ results from averaging the bilocal effective action (5) and performing the limit $T\to\infty$. While details of this calculations are rather involved and will be given elsewhere [14] the final result was already presented in [13]. As $V$ is ultraviolet (UV) divergent we write it down in $d = 4-2\epsilon$ dimensions. In Euclidean time with $p_\mathrm{E}^2 = -M^2$, it takes the form

$$V = V_1 + V_2, \quad (15)$$

$$V_1 = -\frac{\pi\alpha}{\kappa_\mathrm{E}}\,\nu^{2\epsilon}\,(d-1)\int_0^\infty \mathrm{d}\sigma\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{k^2}{k^2+m^2} \times\Big[\big(\dot{\mu}_F^2(\sigma)\big)^2 - \big(\dot{\mu}_B^2(\sigma)\big)^2\Big]\,E(\sigma,k), \quad (16)$$

[1] The integration over the endpoint $x(T) = x$ of the trajectory with weight $\exp(\mathrm{i}p\cdot x)$ (where $p$ is the external momentum of the particle) is included in the average. Note that the present retardation functions have a different normalization from the one in [10].

[2] It should be noted that (8) is still not the most general quadratic ansatz as the retardation functions are assumed to depend only on the proper time difference and additional Lorentz structures are absent. For the general case in a scalar theory see [10], Appendix C.

$$V_2 = \frac{4\pi\alpha}{\kappa_{\rm E}}\,\nu^{2\epsilon}\,\lambda^2 \int_0^\infty {\rm d}\sigma \int \frac{{\rm d}^d k}{(2\pi)^d}\,\frac{1}{k^2+m^2} \times \left[ M^2 - (d-2)\,\frac{(k\cdot p_{\rm E})^2}{k^2} \right] E(\sigma,k)\,. \quad (17)$$

Here $\nu$ is an arbitrary mass parameter, the abbreviation

$$E(k,\sigma) \;=\; \exp\left\{ -\frac{1}{2\kappa_{\rm E}} \left[ k^2 \mu_B^2(\sigma) - 2\lambda k\cdot p_{\rm E}\,\sigma \right] \right\} \quad (18)$$

has been used and a photon mass $m$ has been kept in the photon propagator. The separation of $V$ into two parts makes sense because of several facts: first, it is seen that $V_2$ is more singular than $V_1$, having an additional power $k^2$ in the integrand. Second, $V_2$ vanishes for massless electrons $M = 0$, is repulsive ($V_2 > 0$) and only depends on the bosonic pseudotime.

Previously the anomalous mass dimension $\gamma_M$ was calculated analytically in dimensional regularization [13]. However, from a numerical point of view this regularization is extremely cumbersome in non-perturbative calculations (see, for example, the Dyson–Schwinger equation studies in [15]) and is usually replaced by a simple momentum cut-off. Proper time regularization, i.e. a lower cut-off of proper time integrals at $\sigma = 1/\Lambda^2$, $\Lambda \to \infty$ could also be used (it would maintain translational invariance), but it violates reparametrization invariance.

In the present work, it is much more convenient to make use of a form factor which is very similar to the non-local regularization method proposed in [16]. Here the main reason is that the UV-divergence in $V$ originates in an undamped (for $\sigma \to 0$, in which case $\mu_i^2(\sigma) \to \sigma$) Euclidean momentum integral over the exponential factor $E(k,\sigma)$. This is the only appearance of $\mu_B^2$ in $V$ and hence, after introducing a form factor of the form

$$F(k^2) \;=\; \exp\left( -\frac{k^2}{2\Lambda^2} \right)\,, \quad (19)$$

all dimensionally regulated expressions in [13] may be converted to form factor regulated ones by setting $d = 4$ and making the simple replacement

$$\mu_B^2(\sigma) \;\longrightarrow\; \tilde{\mu}_B^2(\sigma) \;=\; \mu_B^2(\sigma) + \frac{\kappa_{\rm E}}{\Lambda^2} \quad . \quad (20)$$

With this regularization understood, the different terms in the potential $V$ of (15)–(17) become (after performing the momentum integration for $m = 0$)

$$V_1 = -\frac{3\alpha}{4\pi}\kappa_{\rm E} \int_0^\infty {\rm d}\sigma\, \frac{(\dot{\mu}_F^2(\sigma))^2 - (\dot{\mu}_B^2(\sigma))^2}{\tilde{\mu}_B^4(\sigma)}\, {\rm e}^{-\tilde{\gamma}(\sigma)}, \quad (21)$$

$$V_2 = \frac{3\alpha}{\pi}\kappa_{\rm E} \int_0^\infty \frac{{\rm d}\sigma}{\sigma^2}\, \frac{1-[1+\tilde{\gamma}(\sigma)]\,{\rm e}^{-\tilde{\gamma}(\sigma)}}{\tilde{\gamma}(\sigma)}\,. \quad (22)$$

Here

$$\tilde{\gamma}(\sigma) = \frac{\lambda^2 M^2 \sigma^2}{2\kappa_{\rm E}\,\tilde{\mu}_B^2(\sigma)} \quad , \qquad \dot{\mu}^2(\sigma) \equiv \frac{{\rm d}\mu^2(\sigma)}{{\rm d}\sigma} \quad (23)$$

and $\alpha = e^2/(4\pi)$ (in the real world $\simeq 1/137$) is the fine structure constant.

It is clear from (21) and (22) that the $\sigma \to 0$ UV-divergences due to inverse powers of the bosonic pseudotime are now regulated and that both $V_1$ and $V_2$ as well as $\Omega_i$ do not depend on the reparametrization parameter $\kappa_{\rm E}$, which only sets the scale for the $\sigma$- and $E$-variables[3]. One also sees that bosonic and fermionic degrees of freedom do not enter symmetrically in the interaction (e.g. $V_2$ only depends on $\tilde{\mu}_B^2$) and therefore the pseudotimes are in general different. This is due to supersymmetry violation by boundary terms in the trial action and could affect the spin structure of the propagator in our variational scheme. Since the spin-dependent terms are subleading in the limit $T \to \infty$ and are not considered in the present investigation we just let the variational principle decide how much of SUSY violations it tolerates with the present trial action. Fortunately, as we will observe numerically and is to be expected from boundary terms, these supersymmetry violations are restricted to large values of $\sigma$ (or small values of $E$) and therefore do not have any influence on the UV behavior of the solutions.

Because the fermionic contributions, both in the kinetic term of Mano's equation (11) as well as in $V_1$ (see (21)), appear with a sign opposite to the one of the bosonic contributions they would cancel them for exact supersymmetry, i.e. $\mu_F^2(\sigma) = \mu_B^2(\sigma)$. Note that for $\sigma \to 0$ a cancellation of this sort is absolutely necessary as otherwise one would encounter the same quadratic divergences which occur in the propagator of scalar QED. In the context of the variational calculation the cancellation occurs only *after* variation because one needs a restoring force in the variational principle. A simple example illustrating this subtlety is given in [17].

### 2.5 Variational equations

With the explicit expressions for $\Omega$ and $V$ above it is now a simple, albeit somewhat tedious, exercise to derive the variational equations for $\lambda$, $A_F$ and $A_B$. One obtains, respectively,

$$\lambda = 1 - \frac{3\alpha}{2\pi}\frac{\kappa_{\rm E}}{\lambda M^2} \int_0^\infty {\rm d}\sigma \left[ \frac{(\dot{\mu}_F^2)^2 - (\dot{\mu}_B^2)^2}{\tilde{\mu}_B^4}\,\tilde{\gamma}\,{\rm e}^{-\tilde{\gamma}} + \frac{4}{\sigma^2}\frac{(1+\tilde{\gamma}+\tilde{\gamma}^2){\rm e}^{-\tilde{\gamma}}-1}{\tilde{\gamma}} \right], \quad (24)$$

$$1 - \frac{1}{A_i(E)} \;=\; \frac{2}{\kappa_{\rm E}} \int_0^\infty {\rm d}\sigma\; (-)^i\, \frac{\delta V}{\delta \mu_i^2(\sigma)}\, \frac{\sin^2(E\sigma/2)}{E^2 A_i(E)}, \quad (25)$$

where we define $(-)^B = 1$ and $(-)^F = -1$ , the difference in sign originating from the opposite sign of $\Omega[A_B]$ and $\Omega[A_F]$ in Mano's equation.

The functional derivatives in (25) are

$$\frac{\delta V}{\delta \mu_F^2(\sigma)} \;=\; \frac{3\alpha}{2\pi}\kappa_{\rm E}\,\frac{\rm d}{{\rm d}\sigma}\left[ \frac{\dot{\mu}_F^2(\sigma)}{\tilde{\mu}_B^4(\sigma)} {\rm e}^{-\tilde{\gamma}(\sigma)} \right] \quad (26)$$

[3] This is due to the reparametrization dependence of $A_i(E,\kappa_{\rm E}) = A_i(\kappa_{\rm E}E)$ and $\mu_i^2(\sigma,\kappa_{\rm E}) = \kappa_{\rm E}\,\mu_i^2(\sigma/\kappa_{\rm E})$ .

and

$$\frac{\delta V}{\delta\mu_B^2(\sigma)} = -\frac{3\alpha}{2\pi}\kappa_{\rm E}\frac{\rm d}{{\rm d}\sigma}\left[\frac{\dot\mu_B^2(\sigma)}{\tilde\mu_B^4(\sigma)}\,{\rm e}^{-\tilde\gamma(\sigma)}\right] + \frac{3\alpha}{4\pi}\kappa_{\rm E}\frac{1}{\tilde\mu_B^2}\left[\frac{(\dot\mu_F^2)^2-(\dot\mu_B^2)^2}{\tilde\mu_B^4}(2-\tilde\gamma){\rm e}^{-\tilde\gamma} - \frac{4}{\sigma^2}\frac{(1+\tilde\gamma+\tilde\gamma^2)\,{\rm e}^{-\tilde\gamma}-1}{\tilde\gamma}\right]. \quad (27)$$

In the limit in which the interactions are turned off, we have $A_i(E) = \lambda = 1$ and therefore $\mu_i^2(\sigma) = \sigma$, $\Omega_i = 0, M = M_0$. In the interacting theory the pseudotimes $\mu_i^2(\sigma)$ still have linear behavior for both $\sigma \to 0, \infty$. From (26) and (27) we then see that without regularization

$$\frac{\delta V_1}{\delta\mu_i^2(\sigma)} \longrightarrow \frac{\text{const.}}{\sigma^3}, \qquad \frac{\delta V_2}{\delta\mu_B^2(\sigma)} \longrightarrow \frac{\text{const.}}{\sigma^2}, \quad (28)$$

for $\sigma \to 0$. It is this more singular UV-behavior of $\delta V_1$ which requires regularization also in the variational equations. This is the trademark of a renormalizable theory whereas the more benign behavior of $\delta V_2$ is characteristic for a super-renormalizable theory. On the other hand in the infrared region $V_2$ dominates because the very last term in (27) is not exponentially suppressed

$$\frac{\delta V_2}{\delta\mu_B^2(\sigma)} \overset{\sigma\to\infty}{\longrightarrow} \frac{\text{const}}{\sigma^4}. \quad (29)$$

This is due to taking a vanishing photon mass $m = 0$ in (17).

## 3 The variational Abraham–Lorentz-like equations

It is obvious that the profile functions $A_i$ and the pseudotimes $\mu_i^2$ contain the same information, as seen in (13). It would be more efficient, therefore, to eliminate one of these from the variational equations (25). We shall now show that it is possible to rewrite (25) entirely in terms of the pseudotimes.

A differentiation of (13) with respect to $\sigma$ yields

$$\dot\mu_i^2(\sigma) = \frac{\rm d}{{\rm d}\sigma}|\sigma| + \frac{2}{\pi}\int_0^\infty {\rm d}E\frac{\sin(E\sigma)}{E}\left(\frac{1}{A_i(E)}-1\right). \quad (30)$$

It is important here to realize that the pseudotime is even (as can be seen from (13)) and that the first term on the RHS therefore gives sgn $(\sigma)$. The remaining integral is convergent due to subtraction of the asymptotic value $1/A_i(\infty) = 1$. Another differentiation leads to

$$\ddot\mu_i^2(\sigma) = 2\,\delta(\sigma) + \frac{2}{\pi}\int_0^\infty {\rm d}E\cos(E\sigma)\left(\frac{1}{A_i(E)}-1\right). \quad (31)$$

If we now insert (25) we obtain

$$\ddot\mu_i^2(\sigma) = 2\,\delta(\sigma) - \frac{2}{\pi}\frac{1}{\kappa_{\rm E}}\int_0^\infty {\rm d}E\int_0^\infty {\rm d}\sigma'\,(-)^i\frac{\delta V}{\delta\mu_i^2(\sigma')} \times\frac{1}{E^2A_i(E)}\left[1-\cos(E\sigma')\right]\cos(E\sigma). \quad (32)$$

By using the addition theorem for the cosine function and the definition (13) the $E$-integration can be performed exactly and gives

$$\ddot\mu_i^2(\sigma) - \frac{1}{\kappa_{\rm E}}\int_0^\infty {\rm d}\sigma'\,(-)^i\frac{\delta V}{\delta\mu_i^2(\sigma')} \times\left[\mu_i^2(\sigma) - \frac{1}{2}\mu_i^2(\sigma+\sigma') - \frac{1}{2}\mu_i^2(|\sigma-\sigma'|)\right] = 2\,\delta(\sigma),\ \sigma\ge 0. \quad (33)$$

The appropriate boundary conditions for solutions to this integro-differential equation are

$$\mu_i^2(0) = 0, \qquad \lim_{\sigma\to+0}\dot\mu_i^2(\sigma) = 1. \quad (34)$$

The proper times $\sigma, \sigma'$ are restricted to be non-negative and we therefore have to take the absolute value in the argument of the shifted pseudotime in (33). Alternatively, this restriction may be avoided by remembering that $\mu_i^2(\sigma)$ is an even function of $\sigma$ and hence the integrands appearing in the expression for $V$ (see (21) and (22)) are also. One obtains

$$\ddot\mu_i^2(\sigma) - \frac{1}{2\kappa_{\rm E}}\int_{-\infty}^{+\infty}{\rm d}\sigma'\,(-)^i\frac{\delta V}{\delta\mu_i^2(\sigma')}[\mu_i^2(\sigma)-\mu_i^2(\sigma-\sigma')] = 2\,\delta(\sigma),\ -\infty\le\sigma\le+\infty \quad (35)$$

and, in fact, even the $\delta$-function may be eliminated by performing differentiation with respect to $|\sigma|$ rather than $\sigma$:

$$\mu_i^{2\,\prime\prime}(|\sigma|) - \frac{1}{2\kappa_{\rm E}}\int_{-\infty}^{+\infty}{\rm d}\sigma'\,(-)^i\frac{\delta V}{\delta\mu_i^2(|\sigma'|)} \times\left[\mu_i^2(|\sigma|)-\mu_i^2(|\sigma-\sigma'|)\right] = 0, \quad -\infty\le\sigma\le+\infty. \quad (36)$$

Here $''$ denotes differentiation with respect to the argument.

We shall present numerical solutions to these variational equations (which, as promised, no longer involve the profile functions directly) in the next section. At this stage, however, we would first like to discuss the meaning of (36) for the bosonic case, i.e. $i = B$. In [10], (25) and (26), the expectation value of $x^\mu(t_1) - x^\mu(t_2)$ and $[x^\mu(t_1) - x^\mu(t_2)]^2$, when averaged with the trial action, were calculated. The corresponding results for the present case (i.e. QED, with Euclidean metric, and setting $t_2 = 0$ and hence $x^\mu(t_2) = 0$ for convenience) become

$$\begin{aligned}\langle|x(t)|\rangle_{S_{\rm t}} &= \frac{\lambda M}{\kappa_{\rm E}}\,t \;\equiv\; |x_{\rm class.}(t)|,\\ \left\langle x^2(t)\right\rangle_{S_{\rm t}} &= \langle|x(t)|\rangle_{S_{\rm t}}^2 + \frac{4}{\kappa_{\rm E}}\,\mu^2(t)\\ &\equiv x_{\rm class.}^2(t) + \left\langle[\Delta x(t)]^2\right\rangle_{S_{\rm t}},\end{aligned} \quad (37)$$

i.e. the mean squared displacement of the electron is made up by an overall quadratic drift due to the electron's momentum, i.e. growing like $t^2$, and a term proportional to

the pseudotime, growing like $t$ for small $t$; the latter characterizes both the quantum mechanical Brownian motion as well as the continual random "kicks" from emission and absorption of virtual photons in the cloud surrounding the bare particle.

Inserting (37) into the variational equation (36) results in the *Abraham–Lorentz*-like equation mentioned in the Introduction, i.e.

$$\frac{\mathrm{d}^2}{\mathrm{d}|t|^2}\left\langle x^2(t)\right\rangle_{S_\mathrm{t}} = 2\left\langle \dot{x}^2(t)\right\rangle_{S_\mathrm{t}} + \frac{2}{\kappa_\mathrm{E}}\int_{-\infty}^{\infty}\mathrm{d}t'\frac{1}{4}\frac{\delta V}{\delta\mu_i^2(t')} \times\left\{\left\langle\left[\Delta x(t)\right]^2\right\rangle_{S_\mathrm{t}} - \left\langle\left[\Delta x(t-t')\right]^2\right\rangle_{S_\mathrm{t}}\right\}. \quad (38)$$

$\delta V/\delta\mu_i^2$ may then be interpreted as an (internal) "force" acting on the bare electron.

As mentioned in the Introduction, the original Abraham–Lorentz equation actually is a third-order linear differential equation for the position of the electron rather than an integro-differential equation. It is possible to write the VALEs in (33)–(36) in a similar way by converting them into a special form of *delay* equations. This class of differential equations has been studied extensively for problems of radiation damping in electrodynamics and general relativity and in mathematics [18]. Indeed, by invoking the first mean value theorem (this requires that $\delta V/\delta\mu_i^2(\sigma)$ is of one sign, which appears to be fulfilled) to the integral in (36) one can replace the $\sigma'$-argument of the pseudotime by some mean time $0 \le \tau_i \le \infty$. The remaining integral then just gives a constant

$$C_2^i = \int_0^\infty \mathrm{d}\sigma\,(-)^i\frac{\delta V}{\delta\mu_i^2(\sigma)}. \quad (39)$$

Therefore we have for $\sigma \ge 0$

$$\ddot{\mu}_i^2(\sigma) - \frac{C_2^i}{\kappa_\mathrm{E}}\left[\mu_i^2(\sigma) - \frac{1}{2}\mu_i^2\left(\sigma+\tau_i\right) - \frac{1}{2}\mu_i^2\left(|\sigma-\tau_i|\right)\right] = 0. \quad (40)$$

The above (seemingly linear) delay equation is exactly equivalent to the original (non-linear) VALE because the constant $C_2^i$ is in general a functional of the pseudotime and because by construction the delay $\tau_i$ will also depend on the external proper time $\sigma$ as well as on $\mu_i^2(\sigma)$. It is worth noticing that $C_2^i$ also governs the asymptotic behavior of the profile function

$$A_i(E\to\infty) = 1 + \frac{C_2^i}{(\kappa_\mathrm{E}E)^2} + \ldots \quad (41)$$

By differentiating (36) it is seen that it also determines the initial "jerk" [19]:

$$\dddot{\mu}_i^2\,(\sigma\to+0) = \frac{C_2^i}{\kappa_\mathrm{E}}. \quad (42)$$

For the fermionic case (or in the supersymmetric limit) $\delta V/\delta\mu_F^2$ is a total derivative (see (26)), so here one finds the exact expression $C_2^F = 3\alpha\Lambda^4/(2\pi)$.

In many applications of delay equations a constant delay is assumed. Although this does not seem to be a valid approximation for QED it is clear that the delay $\tau_i$ is a remnant of the photon degrees of freedom which have been integrated out and no longer appear in the equation of motion for the mean displacement of the bare electron. Since in QED the "force" $\delta V/\delta\mu_i^2$ is very singular at small $\sigma$ one expects a very small delay $\tau \sim 1/\Lambda^2$ (all integrals are dominated by small values of $\sigma'$): the average time a virtual photon is "in the air" is very short, but the kick given to the bare electron is very violent.

Finally, we emphasize that in the discussion above no use has been made of the specific form of the interaction, i.e. the VALEs (33)–(36) hold for any dressed particle. If we concentrate now on the specific form (21) and (22) of the interaction in QED further simplifications can be made. For investigations of the divergence structure of the theory as in [13] it should be a reasonable approximation to neglect $V_2$ since it was shown to be less singular than $V_1$. Then $\delta V/\delta\mu_i^2(\sigma)$ is just a total derivative (see (26)) and one may integrate the VALE with respect to $\sigma$. In this supersymmetric approximation fermionic and bosonic pseudotimes are identical and for $\sigma \ge 0$ obey the equation

$$\dot{\mu}^2_\mathrm{SUSY}(\sigma) + \frac{3\alpha}{4\pi}\int_0^\infty \mathrm{d}\sigma'\,\frac{\dot{\mu}^2_\mathrm{SUSY}(\sigma')}{\left[\tilde{\mu}^2_\mathrm{SUSY}(\sigma')\right]^2}\,\mathrm{e}^{-\tilde{\gamma}(\sigma')} \times\left[\,\mu^2_\mathrm{SUSY}(\sigma+\sigma') - \mu^2_\mathrm{SUSY}(|\sigma-\sigma'|)\,\right] = 1. \quad (43)$$

Since the regularized pseudotime just adds a constant to the pseudotime $\mu^2(\sigma)$, (43) also holds for $\tilde{\mu}^2(\sigma)$. To exhibit the UV-divergences one may even neglect the electron mass altogether. In this "asymptotic SUSY" (ASUSY) approximation the quantity $\tilde{\gamma}$ in (43) is set to zero which allows for a further integration over $\sigma$. Taking into account the boundary condition $\mu^2(0) = 0$ one obtains the simpler integral equation ($\sigma \ge 0$)

$$\sigma = \mu^2_\mathrm{ASUSY}(\sigma) + \frac{3\alpha}{4\pi}\int_0^\infty \mathrm{d}\sigma'\,\frac{1}{\tilde{\mu}^2_\mathrm{ASUSY}(\sigma')} \times\left[\,\mu^2_\mathrm{ASUSY}(\sigma+\sigma') + \mu^2_\mathrm{ASUSY}(|\sigma-\sigma'|) - 2\mu^2_\mathrm{ASUSY}(\sigma')\,\right]. \quad (44)$$

Adding $\kappa_\mathrm{E}/\Lambda^2$ on both sides turns it into a compact, but still non-linear integral equation for

$$y\left(s = \frac{\sigma\Lambda^2}{\kappa_\mathrm{E}}\right) := \frac{\Lambda^2}{\kappa_\mathrm{E}}\,\tilde{\mu}^2_\mathrm{ASUSY}(\sigma), \quad (45)$$

namely,

$$y(s) = 1 + |s| - \frac{3\alpha}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}s'\,\left[\,\frac{y(s+s')}{y(s')} - 1\right]. \quad (46)$$

We have been unable to find an exact analytical solution for this scale-free equation. For the purpose of calculating the anomalous mass dimension it is, however, sufficient to know the asymptotic behavior of $y(s)$ which will

be derived below. An approximate, third-order differential equation (i.e. of the same type as the original Abraham–Lorentz equation) is obtained in Appendix A.

## 4 Numerical results

Without any approximation the three variational equations, i.e. the fermionic and bosonic VALEs (see (33)) together with the variational equation for $\lambda$ (see (24)) may be solved iteratively by stepwise numerical integration over $\sigma$. Because the profile functions $A_{F,B}$ have been eliminated in favor of the pseudotimes, it is not necessary to numerically evaluate the cosine transform (see (13)) for the pseudotime and its derivative. This simplifies the numerical effort as compared to the published variational treatments of super-renormalizable and finite theories [9, 10,20].

However, the presence of two vastly different scales characterizing both the ultraviolet ( $\sigma \sim \kappa_{\rm E}/\Lambda^2$ ) and the infrared region ( $\sigma \sim \kappa_{\rm E}/(\lambda M^2)$ with $\lambda \to 0$ for $\Lambda \to \infty$, see below) makes the calculation much harder for the present renormalizable theory. Stable numerical results have been obtained for cut-offs as large as $\Lambda = 500\,M$ and coupling constants up to $\alpha = 0.5$–$1.0$ and have been tabulated in Table 1. The cut-off needed to be decreased somewhat as the couplings were increased in order to retain numerical stability. As in previous work (cf. [10,20]), we have checked the accuracy of our results by comparing the direct evaluation of the kinetic terms $\Omega_i$ with those obtained with the help of a virial theorem which relates the $\Omega_i$ to the potential $V$ through the use of the variational equations:

$$\Omega_i|_{\rm var} \;=\; (-)^i \int_0^\infty {\rm d}\sigma \,\left[\,\mu_i^2(\sigma) - \sigma\dot{\mu}_i^2(\sigma)\,\right] \frac{\delta V}{\delta \mu_i^2(\sigma)}\,. \quad (47)$$

A relative accuracy of order $10^{-5}$ was achieved for not too large cut-offs.

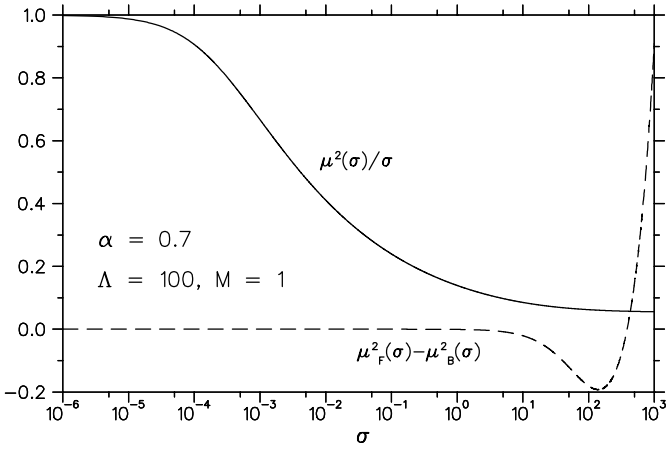


**Fig. 1.** The pseudotime divided by $\sigma$ (solid line) as a function of $\sigma$ (on a logarithmic scale) for coupling constant $\alpha = 0.7$ and cut-off $\Lambda = 100\,M$. Bosonic and fermionic pseudotime are indistinguishable in this plot. The dashed curve shows the difference of both

The results in the table clearly demonstrate, as anticipated in Sect. 2.4, that supersymmetry is almost perfect. This may also be seen in Fig. 1 where the pseudotimes have been plotted for one value of $\alpha$ and $\Lambda$. Differences between the bosonic and fermionic pseudotimes only show up in the infrared region where the less-singular potential contribution $V_2$ dominates. This also leads to slightly different values of the bosonic and fermionic profile function at $E = 0$. Our previous conjecture that SUSY violations would vanish with $\Lambda \to \infty$ [17] turns out to be unfounded: in Table 1 we also give the quantity

$$\Delta_S \;=\; -2\,\frac{\Omega_B - \Omega_F + V_1}{M_0^2}, \quad (48)$$

which may be considered as a measure of the importance of SUSY violations in $M_0^2$. One sees that for the whole range of coupling constants which we consider they remain at the percent-level or below and become cut-off independent within numerical accuracy.

The ratio $\sigma\dot{\mu}^2(\sigma)/\mu^2(\sigma)$ is plotted in Fig. 2 for $\alpha = 0.5$ and various cut-offs. Independently of the actual magnitude of $\mu^2(\sigma)$, power-like behavior of the pseudotime corresponds to a horizontal line on this plot. Because $\mu^2(\sigma) \stackrel{\sigma\to 0}{\longrightarrow} \sigma$ and $\mu^2(\sigma) \stackrel{\sigma\to\infty}{\longrightarrow} \sigma/A(0)$ all curves go towards unity for small and large $\sigma$. What is interesting, however, is that for an increasing range of intermediate $\sigma$-values (as $\Lambda \to \infty$) the pseudotimes exhibit power-like behavior of the form $\sigma^\beta$, $\beta < 1$. Moreover, the value of $\beta$ seems to become independent of the cut-off as $\Lambda$ increases.

This result can be understood analytically: for large cut-offs and moderate values of $\sigma$ the approximate VALEs (43) and (44) should become valid. Inserting the ansatz

$$\tilde{\mu}^2(\sigma) \;\longrightarrow\; s_0 \cdot \sigma^\beta \quad (49)$$

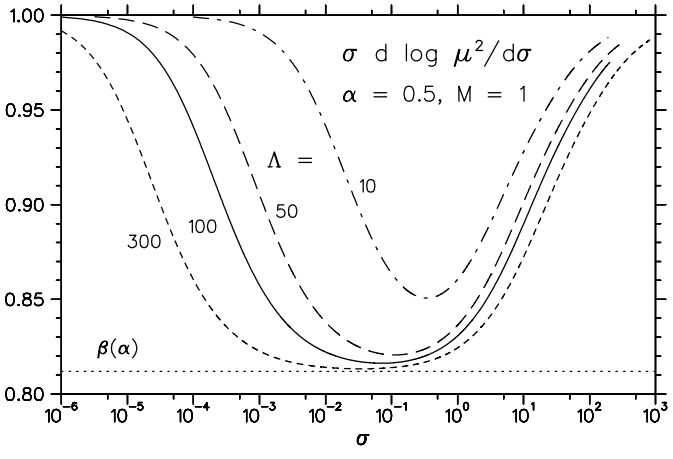


**Fig. 2.** The effective power of $\sigma$ for the bosonic pseudotime $\mu^2(\sigma)$ from numerical solutions of the variational Abraham–Lorentz equation for QED. As in Fig. 1, the difference between $\mu_F^2(\sigma)$ and $\mu_B^2(\sigma)$ is too small to show up on this plot. Results for various cut-offs $\Lambda$ in units of the electron mass $M$ are shown. The dashed horizontal line denotes the numerical value from the analytical solution (52) at coupling constant $\alpha = 0.5$

**Table 1.** Some results of the numerical solution of the variational equations with form factor regularization for different cut-offs and coupling constants. A relative accuracy of $10^{-6}$ was required at each point between subsequent iterations; in entries marked by a $^\star$ only $10^{-5}$ relative accuracy was achieved. The quantity $\Delta_S$ measures the SUSY violation and is defined in (48). The last three columns compare the numerical results with analytical approximations explained in the text

| $\alpha$ | $\Lambda/M$ | $\lambda$ | $A_B(0)$ | $A_F(0)$ | $M_0/M$ | $10^3\Delta_S$ | $\frac{M_0^2}{M^2\lambda}$ | $\beta(\alpha)$ | $\lambda A_B(0)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 50 | 0.69862 | 1.4009 | 1.3992 | 0.81663 | −0.36 | 0.95457 | 0.95450 | 0.97869 |
| | 100 | 0.65402 | 1.4964 | 1.4946 | 0.79011 | −0.37 | 0.95452 | | 0.97866 |
| | 300 | 0.58901 | 1.6615 | 1.6595 | 0.74981 | −0.37 | 0.95451 | | 0.97865 |
| | 500 | 0.56102 | 1.7444 | 1.7423 | 0.73178 | −0.37 | 0.95450 | | 0.97865 |
| | 1000$^\star$ | 0.52515 | 1.8636 | 1.8614 | 0.70800 | −0.34 | 0.9545 | | 0.97865 |
| 0.2 | 50 | 0.48548 | 1.9802 | 1.9716 | 0.66592 | −1.31 | 0.91342 | 0.91332 | 0.96132 |
| | 100 | 0.42570 | 2.2582 | 2.2484 | 0.62355 | −1.38 | 0.91335 | | 0.96129 |
| | 300 | 0.34559 | 2.7815 | 2.7696 | 0.56182 | −1.38 | 0.91333 | | 0.96128 |
| | 500$^\star$ | 0.31366 | 3.0647 | 3.0516 | 0.53522 | −1.42 | 0.9133 | | 0.96128 |
| 0.3 | 50 | 0.33610 | 2.8171 | 2.7923 | 0.54267 | −2.85 | 0.87619 | 0.87610 | 0.94682 |
| | 100 | 0.27631 | 3.4266 | 3.3966 | 0.49201 | −2.88 | 0.87612 | | 0.94681 |
| | 300 | 0.20252 | 4.6752 | 4.6342 | 0.42122 | −2.88 | 0.87610 | | 0.94680 |
| | 500$^\star$ | 0.17527 | 5.402 | 5.355 | 0.39184 | −2.99 | 0.8760 | | 0.94680 |
| 0.4 | 50 | 0.23214 | 4.0256 | 3.9682 | 0.44224 | −4.68 | 0.84249 | 0.84241 | 0.93449 |
| | 100 | 0.17914 | 5.2165 | 5.1423 | 0.38847 | −4.71 | 0.84242 | | 0.93448 |
| | 300$^\star$ | 0.11877 | 7.868 | 7.756 | 0.31629 | −4.78 | 0.8423 | | 0.93448 |
| | 500$^\star$ | 0.09810 | 9.525 | 9.390 | 0.28746 | −4.81 | 0.8423 | | 0.93448 |
| 0.5 | 50 | 0.16015 | 5.769 | 5.651 | 0.36059 | −6.75 | 0.81191 | 0.81186 | 0.92384 |
| | 100 | 0.11616 | 7.953 | 7.791 | 0.30709 | −6.77 | 0.81187 | | 0.92383 |
| | 300$^\star$ | 0.06981 | 13.23 | 12.96 | 0.23805 | −6.88 | 0.8118 | | 0.92383 |
| 0.6 | 50 | 0.11046 | 8.828 | 8.056 | 0.29430 | −8.95 | 0.78410 | 0.78406 | 0.91451 |
| | 100 | 0.07541 | 12.127 | 11.800 | 0.24315 | −8.98 | 0.78406 | | 0.91450 |
| | 300$^\star$ | 0.04117 | 22.21 | 21.61 | 0.17966 | −9.06 | 0.7840 | | 0.91450 |
| 0.7 | 50$^\star$ | 0.07623 | 11.890 | 11.485 | 0.24048 | −11.2 | 0.75867 | 0.75869 | 0.90626 |
| | 100$^\star$ | 0.04905 | 18.48 | 17.85 | 0.19290 | −11.2 | 0.75867 | | 0.90625 |
| 0.8 | 50$^\star$ | 0.05266 | 17.07 | 16.37 | 0.19680 | −13.6 | 0.73540 | 0.73544 | 0.89889 |
| | 100$^\star$ | 0.03198 | 28.10 | 26.95 | 0.15336 | −13.6 | 0.73535 | | 0.89887 |
| 0.9 | 50$^\star$ | 0.03644 | 24.49 | 23.30 | 0.16130 | −15.9 | 0.7140 | 0.71407 | 0.89227 |
| | 100$^\star$ | 0.02091 | 42.66 | 40.60 | 0.12219 | −15.9 | 0.7139 | | 0.89225 |
| 1.0 | 50$^\star$ | 0.02526 | 35.08 | 33.13 | 0.13244 | −18.2 | 0.6943 | 0.69435 | 0.88627 |
| | 100$^\star$ | 0.01372 | 64.61 | 61.01 | 0.09757 | −18.3 | 0.6941 | | 0.88622 |

into (44) one obtains

$$s_0\,\sigma^\beta + \frac{3\alpha}{4\pi}\int_0^\infty \mathrm{d}\sigma'\,\left[\left(1+\frac{\sigma}{\sigma'}\right)^\beta + \left|1-\frac{\sigma}{\sigma'}\right|^\beta - 2\right] = \sigma + \frac{\kappa_\mathrm{E}}{\Lambda^2}\,. \tag{50}$$

Scaling $\sigma' = t\sigma$ shows that the integral is proportional to $\sigma$ and therefore dominates the LHS for large values of $\sigma$ provided $\beta < 1$ which turns out to be the case for *positive* couplings[4]. Hence, for large $\sigma$, the power $\beta$ is only a function of the coupling constant $\alpha$ and determined by the implicit equation

$$\int_0^\infty \mathrm{d}t\,\left[\left(1+\frac{1}{t}\right)^\beta + \left|1-\frac{1}{t}\right|^\beta - 2\right] = \frac{4\pi}{3\alpha}\,. \tag{51}$$

After performing the integral one obtains the following transcendental equation for $\beta(\alpha)$:

$$\frac{\pi}{2}\beta\,\tan\left(\frac{\pi}{2}\beta\right) = \frac{2\pi}{3\alpha}\,. \tag{52}$$

Numerical solutions for a variety of $\alpha$'s have been tabulated in Table 1. For small coupling constants the solution to this equation behaves like

$$\beta \to 1 - \frac{3\alpha}{2\pi} + \left(\frac{3\alpha}{2\pi}\right)^2 + \ldots \tag{53}$$

[4] For $\alpha < 0$ no power-like solutions exist anymore as the term $\sigma^\beta$ would dominate for $\beta > 1$.

confirming that for positive coupling constant $\beta < 1$, while for large $\alpha$ it goes like $\beta \to \sqrt{8/(3\pi\alpha)} \to 0$. Figure 2 shows that the value $\beta(0.5) = 0.81186$ (see Table 1) explains the intermediate-range behavior of the pseudotime very well. For very large $\sigma$ the ASUSY approximation (44) becomes invalid and the pseudotime reverts to linear behavior.

## 5 The anomalous mass dimension

The power-like behavior of $\mu^2(\sigma)$ is also the key for obtaining the variational approximation for the anomalous mass dimension regularized via a form factor rather than dimensionally [13]. To do this we first note that in $V$ the variational parameter $\lambda$ occurs exclusively in the combination $\lambda M$. Let us, therefore, define the alternative dimensionless variational parameter $x_\lambda = \lambda M/\Lambda$ so that Mano's equation becomes

$$M_0^2 \;=\; \Lambda^2 \left[ 2x_\lambda \frac{M}{\Lambda} \;-\; x_\lambda^2 \;-\; 2\,H(\alpha, x_\lambda) \right] \quad , \tag{54}$$

where $H(\alpha, x_\lambda)$ is the dimensionless combination $(\Omega_B - \Omega_F + V)/\Lambda^2$. Note that, by construction, $H(\alpha, x_\lambda)$ no longer depends on $M$ explicitly. Therefore, as it is dimensionless, it can also no longer depend on $\Lambda$ explicitly. The dependence on the cut-off can only enter implicitly through the variational parameters and functions. However, the RHS of (54) is stationary with respect to variational parameters/functions, so that the dependence of $M_0^2$ on $\Lambda$, at fixed $M$, is just given by the explicit $\Lambda$ dependence in (54), i.e.

$$\left.\frac{\partial M_0^2}{\partial \Lambda}\right|_{M \text{ fixed}} \;=\; 2\frac{M_0^2}{\Lambda} - 2M\,x_\lambda\,. \tag{55}$$

Writing $M_0 = Z_M\, M_\nu$ (where $\nu$ is an arbitrary mass parameter), the anomalous mass dimension may now be evaluated with the help of this flow equation as

$$\begin{aligned} \gamma_M = \frac{\partial \log Z_M}{\partial \log \nu} \;&=\; -\frac{\partial \log M_0}{\partial \log \Lambda} \\ &= -\frac{\Lambda}{2M_0^2}\left( 2\frac{M_0^2}{\Lambda} - 2M\,x_\lambda \right) \;=\; \frac{M^2}{M_0^2}\lambda - 1\,. \end{aligned} \tag{56}$$

Of course, the (existence of the) limit $\Lambda \to \infty$ is understood in the above equations. If this limit exists then $\gamma_M$ must necessarily be cut-off independent so that a further differentiation of (56) with respect to $\Lambda$ gives

$$0 \;=\; -\frac{M^2}{M_0^4}\frac{\partial M_0^2}{\partial \Lambda}\lambda + \frac{M^2}{M_0^2}\frac{\partial \lambda}{\partial \Lambda}\,. \tag{57}$$

Using (55) and (56) we obtain $\Lambda\,\partial\lambda/\partial\Lambda = -2\gamma_M\,\lambda$, and integration then shows how the parameter $\lambda$ behaves for very large cut-offs thus:

$$\lambda \;\overset{\Lambda\to\infty}{\longrightarrow}\; \text{const.}\,\left(\frac{M^2}{\Lambda^2}\right)^{\gamma_M} \quad . \tag{58}$$

This behavior comes as no surprise since we know [10] that in the worldline formalism the bare and effective mass of the quantum mechanical particle are $\kappa_\mathrm{E}$ and $\kappa_\mathrm{E}/\lambda$, respectively, for which a similar relation as for the actual masses is expected. Thus the anomalous mass dimension can be determined either numerically by solving the variational equations and evaluating (56) for larger and larger cut-offs or analytically from (58) by finding the cut-off dependence of $\lambda$.

In the following we will pursue the latter option which is possible as we know the approximate behavior of the pseudotime for small, intermediate and large values of $\sigma$ (here we take for simplicity $\kappa_\mathrm{E} = 1$)

$$\tilde{\mu}^2(\sigma) \;\simeq\; \begin{cases} \sigma + \dfrac{1}{\Lambda^2}, & 0 \le \sigma \le \sigma_1, \\ s_0\,\sigma^\beta, & \sigma_1 \le \sigma \le \sigma_2, \\ \dfrac{\sigma}{A(0)}, & \sigma \ge \sigma_2\,. \end{cases} \tag{59}$$

The regions are separated by $\sigma_1 = x_1/\Lambda^2$, where the cut-off $\Lambda^{-2}$ becomes effective in $\tilde{\mu}^2$, and $\sigma_2 = 2x_2/(\lambda^2 M^2 A(0))$ where the exponential $\exp(-\tilde{\gamma})$ becomes important and the ASUSY approximation breaks down. $x_1, x_2$ are numbers of order one which can be determined (together with the normalization factors $A(0)$ and $s_0$) by matching the approximate solutions at the boundaries. As we only need the leading terms in the cut-off $\Lambda$ in order to determine $\gamma_M$, the actual numerical values of $x_1, x_2$ do not matter. Matching gives $s_0 \sim (\Lambda^2)^{\beta-1}$ as expected from dimensional arguments and

$$A(0) \;\sim\; \left(\frac{\Lambda^2}{\lambda^2 M^2}\right)^{\frac{1-\beta}{2-\beta}} \quad . \tag{60}$$

We now insert these expressions into the variational equation for $\lambda$ in SUSY approximation

$$\frac{1}{\lambda} \;=\; 1 + \frac{3\alpha}{\pi}\int_0^\infty \mathrm{d}\sigma\, \frac{1}{\tilde{\mu}^2(\sigma)}\,\frac{(1+\tilde{\gamma}+\tilde{\gamma}^2)\mathrm{e}^{-\tilde{\gamma}} - 1}{\tilde{\gamma}^2}\,. \tag{61}$$

As $\tilde{\gamma}$ only becomes large in the region $\sigma > \sigma_2$ we are allowed to expand the exponential in the intervals for small and medium values of $\sigma$. After performing the integrations, one obtains

$$\begin{aligned} \frac{1}{\lambda} = 1 + \frac{3\alpha}{\pi}\Bigg\{ \frac{1}{2}\ln(1+x_1) \;+\; \frac{1}{2(1-\beta)}\left[ A(0) - \frac{x_1}{1+x_1} \right] \\ +\; A(0)\,\cdot\,\text{const} \Bigg\}\,. \end{aligned} \tag{62}$$

If $A(0)$ would decrease with increasing $\Lambda$ (or stay constant) this equation would imply that $\lambda$ would go to a constant. This, however, is inconsistent with (60). Hence $A(0)$ must increase with $\Lambda$, with (62) implying that $\lambda A(0)$ remains constant. This behavior is clearly seen in the numerical results shown in the last column of Table 1. Also, in this case the low-$\sigma$ region can be neglected, which is reasonable because otherwise the final result would depend on the

details of the regularization. Comparing this with (58) and (60) one sees that consistency requires that

$$\gamma_M = \frac{1-\beta}{\beta}, \tag{63}$$

and so (56) implies that

$$\frac{M_0^2}{M^2\,\lambda} \;=\; \beta \quad . \tag{64}$$

Comparison of the 8th and 9th columns in Table 1 shows clear numerical evidence that this relation is indeed fulfilled. Note also that (52) and (63) imply that $\gamma_M$ is a solution of the implicit equation

$$\frac{3}{4}\alpha \;=\; (1+\gamma_M)\,\tan\left(\frac{\pi}{2}\,\frac{\gamma_M}{1+\gamma_M}\right)\,, \tag{65}$$

in agreement with the result obtained in dimensional regularization [13]. The convergence of the perturbative expansion and the analytic properties of the solutions of such transcendental equations have been studied in [21]. The present derivation adds the insight that for $\alpha < 0$ no power-like solution of the VALE (44) exists and therefore (52) is not applicable anymore. Obviously, this peculiar behavior when changing the sign of the coupling constant is *not* contained in (65) but in agreement with Dyson's old qualitative argument [22] that $\alpha = 0$ is an essential singularity in QED. We also note that the Dyson–Schwinger calculations with a particular ansatz for the electron–photon vertex lead to a similar implicit equation [23], but a gauge dependence remains and solutions exist only below a critical coupling.

## 6 Conclusions and outlook

In summary, we have shown that the variational formulation of worldline QED very naturally leads to an equation which is similar to the one considered much earlier by Abraham, Lorentz and Dirac in attempts to describe the electron and its self-interaction with the radiation field. In contrast to these attempts our approach contains (almost) all the ingredients of the relativistic field theory of electrons and photons, in particular its divergence structure. This has been demonstrated by numerically solving the variational Abraham–Lorentz equation (VALE) for a variety of cut-offs and by deriving an approximate non-perturbative expression for the anomalous mass dimension of the electron. We have shown how the approach leads naturally to a qualitatively different behavior of the theory for $\alpha > 0$ and $\alpha < 0$. Furthermore, while the present investigation has been restricted to a free electron interacting with its own radiation field, the extension to the case when an external field is present as well should be straightforward. This would allow for a study of how this field-theoretically based worldline variational approach to QED avoids the pitfalls of pre-acceleration and acausality which have plagued all classical attempts.

However, for further progress we deem it more important to first study the issue of supersymmetry (breaking) within the worldline variational approach in more detail. For massive electrons small violations of worldline supersymmetry have been observed which are "soft" in the sense that the ultraviolet behavior of the theory is not affected. These violations presumably reflect the SUSY-breaking generated by the different boundary conditions for bosonic and fermionic variables in the exact action. Further investigation is required into what role, if any, this SUSY-breaking plays in the full spin structure of the electron propagator and how it is manifested in the worldline variational approximation. Such an understanding is required for future applications of this non-perturbative approach to physical processes. Finally it is not inconceivable that similar VALE's as derived here in the variational approach for the propagator will also emerge for the full interacting vertex.

## Appendix

## A Approximate third-order differential VALE

Here we show how a third-order differential equation for the mean square displacement arises approximately in QED. For this purpose we invert (13) to obtain

$$\frac{1}{A_i(E)} \;=\; 1 + \int_0^\infty \mathrm{d}\sigma\, \ddot{\mu}_i^2(\sigma)\, \cos(E\sigma) \tag{A.1}$$

and observe that *for small coupling* the profile function stays close to unity. Therefore one has in this case

$$\begin{aligned}\Omega_i &= \frac{2\kappa_\mathrm{E}}{\pi}\int_0^\infty \mathrm{d}E\left[-\log\left(1+\frac{1}{A_i(E)}-1\right)+\frac{1}{A_i(E)}-1\right]\\ &= \frac{\kappa_\mathrm{E}}{\pi}\int_0^\infty \mathrm{d}E\left\{\left[1-\frac{1}{A_i(E)}\right]^2+\mathcal{O}\left(1-\frac{1}{A_i(E)}\right)^3\right\}\\ &\simeq \frac{\kappa_\mathrm{E}}{2}\int_0^\infty \mathrm{d}\sigma\,\left[\,\ddot{\mu}_i^2(\sigma)\,\right]^2\,.\end{aligned} \tag{A.2}$$

Variation first gives a fourth-order ordinary differential equation

$$(-)^i\,\kappa_\mathrm{E}\,\frac{\mathrm{d}^4}{\mathrm{d}\sigma^4}\,\mu_i^2(\sigma) + \frac{\delta V}{\delta \mu_i^2(\sigma)} \;\simeq\; 0\,, \tag{A.3}$$

but in the supersymmetric limit of QED the functional derivative of the "potential" $V$ is a total derivative (see (26)) so that an integration yields

$$\dddot{\mu}^2\,(\sigma) \;\simeq\; \frac{3\alpha}{2\pi}\,\frac{\dot{\mu}^2(\sigma)}{\tilde{\mu}^4(\sigma)}\,\mathrm{e}^{-\tilde{\gamma}(\sigma)}\,. \tag{A.4}$$

This third-order differential equation clearly is specific for QED and therefore it is tempting to interpret the "kinetic" term $\Omega$ in this case as total (integrated) "radiative energy loss". Indeed, it is positive and in the approximate form (A.2) the integrand has precisely the form of the Larmor

power formula (see, e.g. (17.6) in [2]). Without external force there should be, of course, no real radiation loss – in fact, for exact supersymmetry $\Omega_B$ is completely cancelled by $\Omega_F$. Of course, the fourth-order equation also applies (approximately) to other theories where this interpretation does not make sense.

Since we have assumed $\ddot{\mu}^2$ as small in (A.1) the present approximation should be similar to a derivative expansion. Indeed, if in the integrand of the VALE (43),

$$\begin{aligned}\mu^2(\sigma+\sigma') - \mu^2(|\sigma-\sigma'|) \\ = \mu^2(\sigma_>+\sigma_<) - \mu^2(\sigma_>-\sigma_<) \\ \simeq \dot{\mu}^2(\sigma_>)\,\sigma_<,\end{aligned} \tag{A.5}$$

is expanded to first order and the resulting equation divided by $\sigma$ one obtains

$$\begin{aligned}\frac{\dot{\mu}^2(\sigma)}{\sigma}\left[1+\frac{3\alpha}{2\pi}\int_0^\sigma \mathrm{d}\sigma'\,\sigma'\frac{\dot{\mu}^2(\sigma')}{\tilde{\mu}^4(\sigma')}\,\mathrm{e}^{-\tilde{\gamma}(\sigma')}\right] \\ +\frac{3\alpha}{2\pi}\int_\sigma^\infty \mathrm{d}\sigma'\,\frac{\dot{\mu}^4(\sigma')}{\tilde{\mu}^4(\sigma')}\,\mathrm{e}^{-\tilde{\gamma}(\sigma')} \simeq \frac{1}{\sigma}\,.\end{aligned} \tag{A.6}$$

This can be converted to a differential equation by differentiating with respect to $\sigma$ and replacing the square bracket by $1/(\dot{\mu}^2-\sigma\ddot{\mu}^2)$. Another differentiation then gives

$$\dddot{\mu}^2(\sigma) \simeq \frac{3\alpha}{2\pi}\,[\,\dot{\mu}^2(\sigma)-\sigma\ddot{\mu}^2(\sigma)\,]^2\,\frac{\dot{\mu}^2(\sigma)}{\tilde{\mu}^4(\sigma)}\,\mathrm{e}^{-\tilde{\gamma}(\sigma)} \tag{A.7}$$

which is identical with the third-order equation (A.4) if the factor in square brackets is replaced by its perturbative value 1. The boundary conditions are $\mu^2(0)=0\,,\ \dot{\mu}^2(0)=1$ and a peculiar value of the second derivative at the origin

$$\begin{aligned}\ddot{\mu}^2(0) &= \frac{1}{\kappa_\mathrm{E}}\int_0^\infty \mathrm{d}\sigma\,\mu_F^2(\sigma)\,\frac{\delta V}{\delta\mu_F^2(\sigma)}\bigg|_\mathrm{SUSY} \\ &= -\frac{3\alpha}{2\pi}\int_0^\infty \mathrm{d}\sigma\,\left(\frac{\dot{\mu}^2(\sigma)}{\tilde{\mu}^2(\sigma)}\right)^2\,\mathrm{e}^{-\tilde{\gamma}(\sigma)}\end{aligned} \tag{A.8}$$

expressed as functional of the still unknown solution. This follows from (36) in the limit $\sigma\to 0$, together with (26) and can be considered as a reminder that the non-local actions we are working with originate from local actions with the photonic degrees of freedom integrated out. In this way Ostrogradski's no-go theorem for higher derivative and non-local theories [24] is evaded.

It is interesting that the (approximate) equation (A.7) also allows for power-like solutions in the massless case whereas the perturbative version (A.4) does not. Indeed, inserting the ansatz (49) into (A.7) shows that the powers on both sides of the equation match and this leads to the cubic equation

$$\beta-1 \simeq \frac{3\alpha}{2\pi}\,\beta^2(\beta-2)\,. \tag{A.9}$$

This gives the correct limit (53) for $\alpha\to 0$ and the same $1/\sqrt{\alpha}$-behavior for large $\alpha$ as the exact result (52) only with a coefficient $\pi/\sqrt{8}=1.11$ times larger. That the derivative expansion works rather well also at intermediate coupling constants is demonstrated, for example, by the numerical value $\beta(0.5)\simeq 0.81277$ obtained from (A.9) to be compared with the correct $\beta(0.5)=0.81186$.

## References


1. S.S. Schweber, QED and the men who made it (Princeton University Press, Princeton 1994)
2. J.D. Jackson, Classical electrodynamics, 2nd edn. (John Wiley, New York 1975), chapter 17
3. P.A.M. Dirac, Proc. Roy. Soc. A **167**, 148 (1938)
4. M. Ibison, H.E. Puthoff, J. Phys. A **34**, 3421 (2001)
5. H. Levine, E.J. Moniz, D.H. Sharp, Am. J. Phys. **45**, 75 (1977); E.J. Moniz, D.H. Sharp, Phys. Rev. D **15**, 2850 (1977)
6. P.R. Johnson, B.L. Hu, Phys. Rev. D **65**, 0655015 (2002); quant-ph 0012137
7. R.P. Feynman, Phys. Rev. **97**, 660 (1955)
8. K. Mano, Progr. Theor. Phys. **14**, 435 (1955)
9. R. Rosenfelder, A.W. Schreiber, Phys. Rev. D **53**, 3337 (1996); D **53**, 3354 (1996); A.W. Schreiber, R. Rosenfelder, C. Alexandrou, Int. J. Phys. E **5**, 681 (1996); A.W. Schreiber, R. Rosenfelder, Nucl. Phys. A **601**, 397 (1996); C. Alexandrou, R. Rosenfelder, A.W. Schreiber, Nucl. Phys. A **628**, 427 (1998); N. Fettes, R. Rosenfelder, Few-Body Syst. **24**, 1 (1998)
10. R. Rosenfelder, A.W. Schreiber, Eur. Phys. J. C **25**, 139 (2002)
11. See, e.g. M.J. Strassler, Nucl. Phys. B **385**, 145 (1992); M. Reuter, M.G. Schmidt, Ch. Schubert, Ann. Phys. **259**, 313 (1997); Ch. Schubert, Phys. Rep. **355**, 73 (2001)
12. C. Alexandrou, R. Rosenfelder, A.W. Schreiber, Phys. Rev. A **59**, 1762 (1999)
13. C. Alexandrou, R. Rosenfelder, A.W. Schreiber, Phys. Rev. D **62**, 085009 (2000)
14. R. Rosenfelder, A.W. Schreiber, to be published
15. A.W. Schreiber, T. Sizer, A.G. Williams, Phys. Rev. D **58**, 125014 (1998)
16. D. Evans, J.W. Moffat, G. Kleppe, R.P. Woodard, Phys. Rev. D **43**, 499 (1991)
17. A.W. Schreiber, R. Rosenfelder, C. Alexandrou, in Proceedings of the Workshop on Light-Cone QCD and Nonperturbative Hadron Physics, Adelaide (Australia), 13–22 December 1999, edited by A.W. Schreiber, A.G. Williams (World Scientific, Singapore 2000), p. 295
18. C. Chicone, S.M. Kopeikin, B. Mashhoon, D.G. Retzloff, Phys. Lett. A **285**, 17 (2001); D. Zwilliger, Handbook of differential equations (Academic Press, New York 1989), chapter 42
19. M.W. Leen, Am. J. Phys. **62**, 393 (1994)
20. R. Rosenfelder, A.W. Schreiber, Phys. Lett. A **284**, 63 (2001)
21. V.E. Markushin, R. Rosenfelder, A.W. Schreiber, Nuovo Cim. B **117**, 75 (2002)
22. F.J. Dyson, Phys. Rev. **85**, 631 (1952)
23. A. Kizilersü, A.W. Schreiber, A.G. Williams, Phys. Lett. B **499**, 261 (2001)
24. D.L. Bennet, H.B. Nielsen, R.P. Woodard, Phys. Rev. D **57**, 1167 (1998); R.P. Woodard, Phys. Rev. A **67**, 016102 (2003)